**Example 1**

Evaluate each trigonometric sum or difference.

<table>
<thead>
<tr>
<th>Sum and Difference Identities</th>
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<tr>
<td>( \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B )</td>
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a) \( \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \)

b) \( \sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \)

c) \( \cos(45^\circ - 60^\circ) = \)

d) \( \cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \)

e) \( \tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \)

f) \( \tan\left(\frac{\pi}{6} - \frac{\pi}{3}\right) = \)
Example 2 Write each expression as a single trigonometric ratio.

a) \( \sin \frac{\pi}{6} \cos \frac{\pi}{2} + \cos \frac{\pi}{6} \sin \frac{\pi}{2} \)

b) \( \frac{\tan \frac{\pi}{4} - \tan \frac{\pi}{6}}{1 + \tan \frac{\pi}{4} \tan \frac{\pi}{6}} \)

c) \( \cos \frac{\pi}{3} \cos \frac{\pi}{6} + \sin \frac{\pi}{3} \sin \frac{\pi}{6} \)

Sum and Difference Identities

- \( \sin (A \pm B) = \sin A \cos B \pm \cos A \sin B \)
- \( \cos (A \pm B) = \cos A \cos B \mp \sin A \sin B \)
- \( \tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \)
Lesson Notes

Example 3  Find the exact value of each expression.

a) \( \cos 15° \)

b) \( \sin \frac{5\pi}{12} \)

c) \( \tan 195° \)

d) Given the exact values of cosine and sine for 15°, fill in the blanks for the other angles.

\[
P(15°) = \left( \frac{\sqrt{6} + \sqrt{2}}{4}, \frac{\sqrt{6} - \sqrt{2}}{4} \right)
\]

Sum and Difference Identities

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Find the exact value of each expression. For simplicity, do not rationalize the denominator.

Example 4

a) \( \csc\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \)

b) \( \sec\left(\frac{\pi}{12}\right) \)

c) \( \cot\left(\frac{\pi}{2} - \frac{\pi}{4}\right) \)

Sum and Difference Identities

- \( \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \)
- \( \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \)
- \( \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \)
Example 5
Double-angle identities.

a) Prove the double-angle sine identity, \( \sin 2x = 2 \sin x \cos x \).

b) Prove the double-angle cosine identity, \( \cos 2x = \cos^2 x - \sin^2 x \).

c) The double-angle cosine identity, \( \cos 2x = \cos^2 x - \sin^2 x \), can be expressed as \( \cos 2x = 1 - 2 \sin^2 x \) or \( \cos 2x = 2 \cos^2 x - 1 \). Derive each identity.

d) Derive the double-angle tan identity, \( \tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \).
Example 6 Double-angle identities.

a) Evaluate each of the following expressions using a double-angle identity.

i) \( \sin 60^\circ \)  
ii) \( \cos \frac{\pi}{2} \)  
iii) \( \tan 90^\circ \)

b) Express each of the following expressions using a double-angle identity.

i) \( \sin 8x \)  
ii) \( \cos 4x \)  
iii) \( \sin x \)  
iv) \( \cos \frac{1}{2}x \)

c) Write each of the following expression as a single trigonometric ratio using a double-angle identity.

i) \( \cos^2 30^\circ - \sin^2 30^\circ \)  
ii) \( \sin \frac{\pi}{8} \cos \frac{\pi}{8} \)  
iii) \( 1 - \sin^2 \frac{1}{2}x \)  
iv) \( \frac{2 \tan \frac{x}{8}}{1 - \tan^2 \frac{x}{8}} \)
Example 7  Prove each trigonometric identity.
Note: Variable restrictions may be ignored for the proofs in this lesson.

a) \( \cos\left(\frac{x + \frac{5\pi}{6}}{6}\right) = -\frac{\sqrt{3}}{2} \cos x + \frac{\sin x}{2} \)

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b) \( \sin\left(\frac{3\pi}{2} - x\right) = -\cos x \)

c) \( \tan\left(\frac{x - \frac{3\pi}{4}}{4}\right) = \frac{\tan x + 1}{1 - \tan x} \)

d) \( \cos(x + y) + \cos(x - y) = 2 \cos x \cos y \)
Example 8  Prove each trigonometric identity.

a) \( \cos\left(x + \frac{\pi}{6}\right) - \sin\left(x + \frac{2\pi}{3}\right) = 0 \)

b) \( \frac{\sin(x - y)}{\cos x \cos y} = \tan x - \tan y \)

c) \( \cos(x + y) \cos(x - y) = (\cos x \cos y)^2 - (\sin x \sin y)^2 \)

d) \( \cos 2x = \cos^2 x - \sin^2 x \)

Sum and Difference Identities

\[
\begin{align*}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{align*}
\]
Example 9  Prove each trigonometric identity.

a) \( \cos 2x + 2\sin^2 x = 1 \)

b) \( \frac{2}{1 + \cos 2x} = \sec^2 x \)

c) \( \frac{\sin 2x}{\cos 2x + \sin^2 x} = 2 \tan x \)

d) \( \frac{2\sin x \cos x}{\cos^2 x - \sin^2 x} = \tan 2x \)

Double-Angle Identities

- \( \sin 2x = 2\sin x \cos x \)
- \( \cos 2x = \cos^2 x - \sin^2 x \)
- \( \cos 2x = 2\cos^2 x - 1 \)
- \( \cos 2x = 1 - 2\sin^2 x \)
- \( \tan 2x = \frac{2\tan x}{1 - \tan^2 x} \)
Example 10 Prove each trigonometric identity.

a) \( \cos^4 x - \sin^4 x = \cos 2x \)

b) \( 1 - (\sin x + \cos x)^2 = -\sin 2x \)

c) \( \frac{2(\tan x - \cot x)}{\tan^2 x - \cot^2 x} = \sin 2x \)

d) \( \frac{1}{1 - \tan x} - \frac{1}{1 + \tan x} = \tan 2x \)

Double-Angle Identities:

- \( \sin 2x = 2 \sin x \cos x \)
- \( \cos 2x = \cos^2 x - \sin^2 x \)
- \( \cos 2x = 2 \cos^2 x - 1 \)
- \( \cos 2x = 1 - 2 \sin^2 x \)
- \( \tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \)
Example 11  Prove each trigonometric identity.

a) \[ 2 \csc 2x = \csc x \sec x \]

b) \[ \frac{\sin(x + y)}{\cos x \sin y} = \tan x \cot y + 1 \]

c) \[ \sin 88^\circ \cos 58^\circ - \cos 88^\circ \sin 58^\circ = \frac{1}{2} \]

d) \[ \tan \left( x + \frac{\pi}{4} \right) = \frac{\tan x + 1}{1 - \tan x} \]
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Example 12  Prove each trigonometric identity.

a) \((\sin x + \cos x)^2 - 1 = \sin 2x\) 

b) \(\frac{1}{2} \sin \frac{2x}{5} = \sin \frac{x}{5} \cos \frac{x}{5}\)

c) \(\cos^2 \left(x - \frac{\pi}{2}\right) = \sin^2 x\) 

d) \(\sin 3x = 3 \sin x - 4 \sin^3 x\)
Example 13  Prove each trigonometric identity.

a) \[ \frac{5 \sin x - \cos 2x - 11}{2 \sin x - 3} = \sin x + 4 \]

b) \[ \cos 3x = 4 \cos^3 x - 3 \cos x \]

c) \[ \cos 34^\circ \cos 41^\circ - \sin 34^\circ \sin 41^\circ = \frac{\sqrt{6} - \sqrt{2}}{4} \]

d) \[ \frac{\tan x + \tan y}{\sec x \sec y} = \sin (x + y) \]
Example 14 Solve each trigonometric equation over the domain $0 \leq x \leq 2\pi$.

a) $\cos 2x = \cos^2 x$

b) $\cos \left( x + \frac{\pi}{4} \right) + \cos \left( x - \frac{\pi}{4} \right) = -1$

c) $4 \sin^2 x + 4 \cos 2x - 1 = 0$

d) $2 \cos^2 \frac{1}{2} x - 2 \sin^2 \frac{1}{2} x = 1$
Example 15  Solve each trigonometric equation over the domain $0 \leq x \leq 2\pi$.

a) $\cos 2x + 7 \sin x - 4 = 0$

b) $\sin 2x - \cos x = 0$

c) $\sin \left(\frac{\pi}{3} + x\right) - \sin \left(\frac{\pi}{3} - x\right) = 1$

d) $\sin x \cos x = \frac{1}{4}$
Example 16  Solve each trigonometric equation over the domain $0 \leq x \leq 2\pi$.

a) $\cos 2x - \cos x = 0$

b) $\csc \left( x + \frac{\pi}{2} \right) - \csc \left( x - \frac{\pi}{2} \right) = 4$

c) $\frac{1}{2} \sin 2x + \sin x = 0$

d) $2 \cot^2 x - 3 \csc x = 0$
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Example 17

Solve each trigonometric equation over the domain $0 \leq x \leq 2\pi$.

a) $8 \sin x \cos x = 2$

b) $(\cos x - \sin x)^2 = \sin 2x + 1$

c) $\tan(x - \pi) + \sec x = 0$

d) $\cos(x + \pi) - \cos^2 x = 0$
Example 18  Trigonometric identities and geometry.

a) Show that $\tan B = \frac{\tan A + \tan C}{1 - \tan A \tan C}$

b) If $A = 32^\circ$ and $B = 89^\circ$, what is the value of $C$?
Example 19  
Trigonometric identities and geometry.

Solve for x. Round your answer to the nearest tenth.

\[
\begin{align*}
\cos 15^\circ &= \frac{\sqrt{6} + \sqrt{2}}{4} \\
\sin 15^\circ &= \frac{\sqrt{6} - \sqrt{2}}{4}
\end{align*}
\]
If a cannon shoots a cannonball \( \theta \) degrees above the horizontal, the horizontal distance traveled by the cannonball before it hits the ground can be found with the function:

\[
d(\theta) = \frac{v_i^2 \sin \theta \cos \theta}{4.9}
\]

The initial velocity of the cannonball is 36 m/s.

a) Rewrite the function so it involves a single trigonometric identity.

b) Graph the function. Use the graph to describe the trajectory of the cannonball at the following angles: 0°, 45°, and 90°.

c) If the cannonball travels a horizontal distance of 100 m, find the angle of the cannon. Solve graphically, and round your answer to the nearest tenth of a degree.
Example 21

An engineer is planning the construction of a road through a tunnel. In one possible design, the width of the road maximizes the area of a rectangle inscribed within the cross-section of the tunnel.

The angle of elevation from the centre line of the road to the upper corner of the rectangle is $\theta$. Sidewalks on either side of the road are included in the design.

a) If the area of the rectangle can be represented by the function $A(\theta) = m \sin 2\theta$, what is the value of $m$?

b) What angle maximizes the area of the rectangular cross-section?

c) For the angle that maximizes the area:
   i) What is the width of the road?
   ii) What is the height of the tallest vehicle that will pass through the tunnel?
   iii) What is the width of one of the sidewalks? Express answers as exact values.
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Example 22

The improper placement of speakers for a home theater system may result in a diminished sound quality at the primary viewing area. This phenomenon occurs because sound waves interact with each other in a process called interference. When two sound waves undergo interference, they combine to form a resultant sound wave that has an amplitude equal to the sum of the component sound wave amplitudes.

If the amplitude of the resultant wave is larger than the component wave amplitudes, we say the component waves experienced constructive interference.

If the amplitude of the resultant wave is smaller than the component wave amplitudes, we say the component waves experienced destructive interference.

a) Two sound waves are represented with \( f(\theta) \) and \( g(\theta) \).

i) Draw the graph of \( y = f(\theta) + g(\theta) \) and determine the resultant wave function.
ii) Is this constructive or destructive interference?
iii) Will the new sound be louder or quieter than the original sound?
b) A different set of sound waves are represented with \( m(\theta) \) and \( n(\theta) \).

i) Draw the graph of \( y = m(\theta) + n(\theta) \) and determine the resultant wave function.

ii) Is this constructive or destructive interference?

iii) Will the new sound be louder or quieter than the original sound?

\[
m(\theta) = 2\cos \theta \\
n(\theta) = 2\cos(\theta - \pi)
\]

c) Two sound waves experience total destructive interference if the sum of their wave functions is zero. Given \( p(\theta) = \sin(3\theta - 3\pi/4) \) and \( q(\theta) = \sin(3\theta - 7\pi/4) \), show that these waves experience total destructive interference.
Lesson Notes

Example 23  Even & Odd Identities

a) Explain what is meant by the terms *even function* and *odd function*.

Even & Odd Identities

\[
\begin{align*}
\sin(-x) &= -\sin x \\
\cos(-x) &= \cos x \\
\tan(-x) &= -\tan x 
\end{align*}
\]

b) Explain how the even & odd identities work.

*(Reference the unit circle or trigonometric graphs in your answer.)*

c) Prove the three even & odd identities algebraically.
Example 24 Proving the sum and difference identities.

a) Explain how to construct the diagram shown.

b) Explain the next steps in the construction.

Enrichment Example
Students who plan on taking university calculus should complete this example.
c) State the side lengths of all the triangles.

\[
\alpha + \beta
\]

\[
\alpha
\]

\[
\beta
\]

d) Prove the sum and difference identity for sine.

\[
\cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}
\]

\[
\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}
\]